

GATEAEROSPACEFORUM GATE Classes Lecture Notes :

Class : Numerical Method (Topic : Secant Method)

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THE ORDER OF CONVERGENCE FOR THE SECANT METHOD.

Suppose that we are solving the equation $f(x) = 0$ using the secant method. Let the iterations

$$(1) \quad x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, 3, \dots,$$

be successful and approach a solution α , $f(\alpha) = 0$, as $n \rightarrow \infty$. How fast do they converge? Can we find the exponent p such that

$$|x_{n+1} - \alpha| \approx C|x_n - \alpha|^p,$$

as we did for bisections and Newton's method?

Yes we can, but the error analysis is a bit more involved.

Equation (1) expresses the iterate x_{n+1} as a function of x_n and x_{n-1} .

Let $x_n = \alpha + \epsilon_n$. Since $x_n \rightarrow \alpha$, the sequence of errors ϵ_n approaches 0 as $n \rightarrow \infty$. In terms of α and ϵ_n the formula becomes

$$(2) \quad \epsilon_{n+1} = \epsilon_n - \frac{f(\alpha + \epsilon_n)(\epsilon_n - \epsilon_{n-1})}{f(\alpha + \epsilon_n) - f(\alpha + \epsilon_{n-1})}.$$

Assume that $f(x)$ is a two times differentiable function and that $f'(\alpha), f''(\alpha) \neq 0$.
Write Taylor's formula:

$$f(\alpha + \epsilon) = f(\alpha) + f'(\alpha)\epsilon + \frac{f''(\alpha)}{2} \epsilon^2 + R_2(\epsilon).$$

Here $f(\alpha) = 0$, ϵ is small, and $R_2(\epsilon)$ is the remainder term. Recall that $R_2(\epsilon)$ vanishes at $\epsilon = 0$ at a faster rate than ϵ^2 . Neglecting the terms of order higher than 2 in ϵ , we have the approximation

$$f(\alpha + \epsilon) \approx f'(\alpha)\epsilon + \frac{f''(\alpha)}{2} \epsilon^2.$$

Set for brevity

$$M = \frac{f''(\alpha)}{2f'(\alpha)},$$

and use the approximate equalities

$$\begin{aligned} f(\alpha + \epsilon_n) &\approx \epsilon_n f'(\alpha)(1 + M\epsilon_n) \\ f(\alpha + \epsilon_n) - f(\alpha + \epsilon_{n-1}) &\approx f'(\alpha)(\epsilon_n - \epsilon_{n-1})(1 + M(\epsilon_n + \epsilon_{n-1})) \end{aligned}$$

to simplify equation (2):

$$\begin{aligned} \epsilon_{n+1} &\approx \epsilon_n - \frac{\epsilon_n f'(\alpha)(1 + M\epsilon_n)(\epsilon_n - \epsilon_{n-1})}{f'(\alpha)(\epsilon_n - \epsilon_{n-1})(1 + M(\epsilon_n + \epsilon_{n-1}))} \\ &= \epsilon_n - \frac{\epsilon_n(1 + M\epsilon_n)}{1 + M(\epsilon_n + \epsilon_{n-1})} \\ &= \frac{\epsilon_{n-1}\epsilon_n M}{1 + M(\epsilon_n + \epsilon_{n-1})} \\ &\approx \epsilon_{n-1}\epsilon_n M. \end{aligned}$$

We have obtained a relation for the errors,

$$(3) \quad \epsilon_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)} \epsilon_{n-1} \epsilon_n,$$

where the terms of order higher than 2 in ϵ are neglected.

Compare (3) to the corresponding formula for Newton's method:

$$\epsilon_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)} \epsilon_n^2.$$

Formula (3) tells us that, as $n \rightarrow \infty$, the error tends to zero faster than a linear function and yet not quadratically. What exactly is the rule

$$|\epsilon_{n+1}| \approx C |\epsilon_n|^p$$

determined by (3)? Argue as follows. If $|\epsilon_{n+1}| \approx C |\epsilon_n|^p$ then

$$C |\epsilon_n|^p \approx |M| |\epsilon_{n-1}| |\epsilon_n|$$

$$|\epsilon_n|^{p-1} \approx \frac{|M|}{C} |\epsilon_{n-1}|$$

$$|\epsilon_n| \approx \left(\frac{|M|}{C} \right)^{\frac{1}{p-1}} |\epsilon_{n-1}|^{\frac{1}{p-1}}.$$

Therefore, $C = \left(\frac{|M|}{C} \right)^{\frac{1}{p-1}}$ and $p = \frac{1}{p-1}$. Because $p > 0$, the condition on p gives

$$p = \frac{1 + \sqrt{5}}{2} \approx 1.618.$$

The condition on C then implies that

$$C^p = |M| \quad \text{or} \quad C = |M|^{1/p} = \left| \frac{f''(\alpha)}{2f'(\alpha)} \right|^{p-1}.$$

We conclude that for the secant method

$$|x_{n+1} - \alpha| \approx \left| \frac{f''(\alpha)}{2f'(\alpha)} \right|^{\frac{\sqrt{5}-1}{2}} |x_n - \alpha|^{\frac{\sqrt{5}+1}{2}}.$$

Evidently, the order of convergence is generally lower than for Newton's method. However the derivatives $f'(x_n)$ need not be evaluated, and this is a definite computational advantage.